INSTITUT FÜR
INFORMATIK
FREIBURG

Theory I
Algorithm Design and Analysis

(5 Hashing)

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The dictionary problem

Different approaches to the dictionary problem:

- Previously: Structuring the set of actually occurring keys: lists, trees, graphs, ...
- Structuring the complete universe of all possible keys: hashing

Hashing describes a special way of storing the elements of a set by breaking down the universe of possible keys.

The position of the data element in the memory is given by computation directly from the key.

Hashing

Dictionary problem:

Lookup, insertion, deletion of data sets (keys)

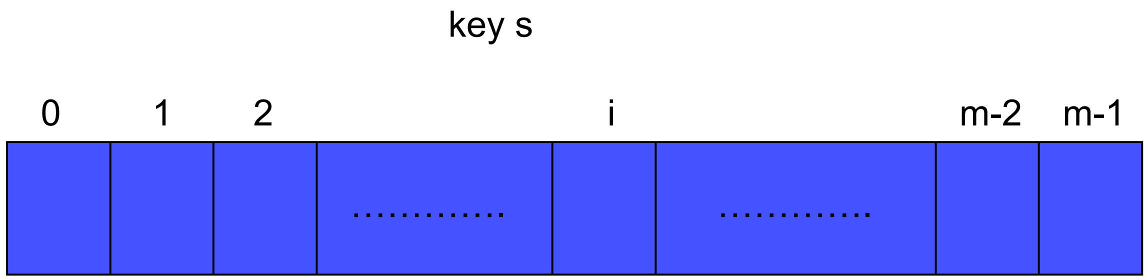
Place of data set *d*: computed from the key *s* of *d*

→ no comparisons

→ constant time

Data structure: linear field (array) of size *m*

Hash table



The memory is divided in *m* containers (buckets) of the same size.

Hash tables - examples

Examples:

- Compilers
  i int 0x87C50FA4
  j int 0x87C50FA8
  x double 0x87C50FAC
  name String 0x87C50FB2
  ...
- Environment variables (key, attribute) list
  EDITOR=emacs
  GROUP=mitarbeiter
  HOST=vulcano
  HOSTTYPE=sun4
  LPDEST=hp5
  MACHTYPE=sparc
  ...
- Executable programs
  PATH=˜/bin:/usr/local/gnu/bin:/usr/local/bin:/usr/bin:/bin:

Implementation in Java

```
class TableEntry {
    private Object key,value;
}

abstract class HashTable {
    private TableEntry[] tableEntry;
    private int capacity;

    // Construktor
    HashTable (int capacity) {
        this.capacity = capacity;
        tableEntry = new TableEntry [capacity];
        for (int i = 0; i <= capacity-1; i++)
            tableEntry[i] = null;
    }
    // the hash function
    protected abstract int h (Object key);

    // insert element with given key and value (if not there already)
    public abstract void insert (Object key Object value);

    // delete element with given key (if there)
    public abstract void delete (Object key);

    // locate element with given key
    public abstract Object search (Object key);
} // class hashTable
```

Hashing - problems

1. Size of the hash table

   Only a small subset *S* of all possible keys (the universe) *U* actually occurs

2. Calculation of the address of a data set

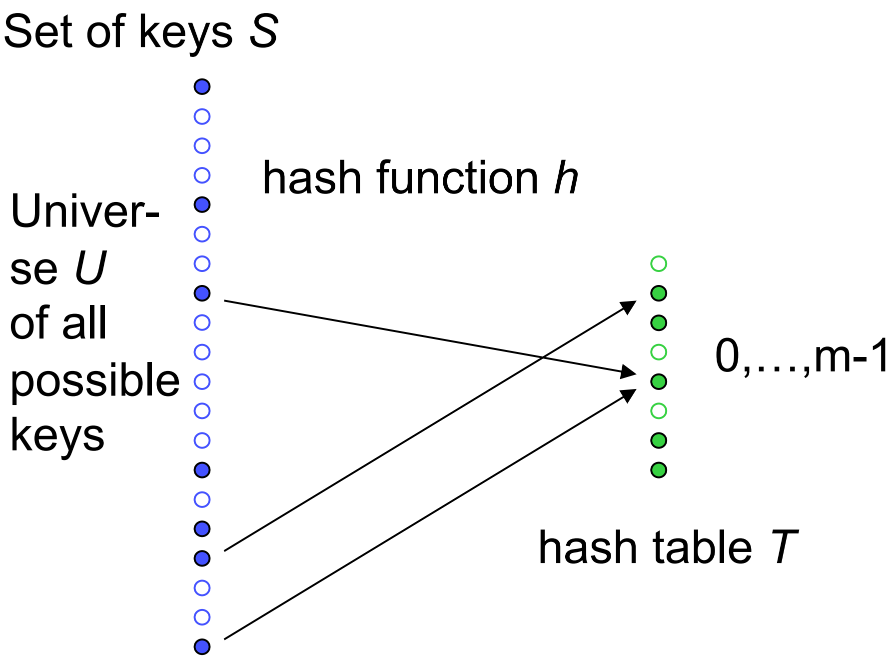
   - keys are not necessarily integers
   - index depends on the size of hash table

In Java:

```
public class Object {
     ...
     public int hashCode()  {…}
     ...
}
```

The universe *U* should be distributed as evenly as possibly to the numbers $-2^{31}, \ldots, 2^{31}-1$.

Hash function (1)



$$(H(U) \subseteq [-2^{31}, 2^{31}-1])$$

$h(s)$ = hash address

$h(s) = h(s')$ $\Leftrightarrow$ s and s' are synonyms with respect to h

address collision

Hash function (2)

Definition: Let U be a universe of possible keys and $\{B_0, . . . , B_{m-1}\}$ a set of m buckets for storing elements from U. Then a hash function is a total mapping

$$h : U \rightarrow \{0, \ldots , m - 1\}$$

mapping each key $s \in U$ to a number $h(s)$
(and the respective element to the bucket $B_{h(s)}$).

- The bucket numbers are also called hash addresses, the complete set of buckets is called hash table.

B_0
B_1
…
…
B_{m-1}

Address collisions

- A hash function *h* calculates for each key *s* the number of the associated bucket.
- It would be ideal if the mapping of a data set with key *s* to a bucket *h*(*s*) was unique (one-to-one): insertion and lookup could be carried out in constant time ($O(1)$).
- In reality, there will be collisions: several elements can be mapped to the same hash address. Collisions have to be treated (in one way or another).

Hashing methods

Example for U: all names in Java with length ≤ 40 → $|U| = 62^{40}$

If $|U| > m$: address collisions are inevitable

Hashing methods:

1. Choice of a hash function that is as "good" as possible
2. Strategy for resolving address collisions

Load factor α :

$$\alpha = \frac{\#\ \text{stored keys}}{\text{size of the hash table}} = \frac{|S|}{m} = \frac{n}{m}$$

Assumption: table size m is fixed

Requirements for good hash functions

Requirements

- A collision occurs if the bucket $B_{h(s)}$ for a newly inserted element with key s is already taken.
- A hash function h is called perfect for a set S of keys if no collisions will occur for S.
- If h is perfect and $|S| = n$, then $n \leq m$.
  The load factor of the hash table is $n/m \leq 1$.
- A hash function is well chosen if

  – the load factor is as high as possible,
  – for many sets of keys the # of collisions is as small as possible,
  – it can be computed efficiently.

Example of a hash function

Example: hash function for strings

```
public static int h (String s){
    int k = 0, m = 13;
    for (int i=0; i < s.length(); i++)
        k += (int)s.charAt (i);
    return ( k%m );
}
```

The following hash addresses are generated for $m = 13$.

key s	$h(s)$
Test	0
Hallo	2
SE	9
Algo	10

The greater the choice of m, the more perfect h becomes.

Probability of collision (1)

Choice of the hash function

- The requirements high load factor and small number of collisions are in conflict with each other. We need to find a suitable compromise.
- For the set S of keys with $|S| = n$ and buckets $B_0, ..., B_{m-1}$:
  – for $n > m$ conflicts are inevitable
  – for $n < m$ there is a (residual) probability $P_K(n,m)$ for the occurrence of at least one collision.

How can we find an estimate for $P_K(n,m)$?

- For any key s the probability that $h(s) = j$ with $j \in \{0, ..., m - 1\}$ is:
  $P_K\,[h(s) = j] = 1/m$, provided that there is an equal distribution.
- We have $P_K(n,m) = 1 - P_{\neg K}(n,m)$,
  if $P_{\neg K}(n,m)$ is the probability that storing of n elements in m buckets leads to no collision.

Probability of collision (2)

On the probability of collisions

- If n keys are distributed sequentially to the buckets B_0, ..., B_{m-1} (with equal distribution), each time we have $P\,[h(s) = j\,] = 1/m$.
- The probability $P(i)$ for no collision in step i is $P(i) = (m - (i - 1))/m$
- Hence, we have

$$P_K(n,m) = 1 - P(1) * P(2) * \ldots * P(n) = 1 - \frac{m(m-1)\ldots(m-n+1)}{m^n}$$

For example, if $m = 365$, $P(23) > 50\%$ and $P(50) \approx 97\%$ (“birthday paradox”)

Common hash functions

Hash fuctions used in practice:

- see: D.E. Knuth: *The Art of Computer Programming*
- For U = integer the [divisions-residue method] is used:

  $h(s) = (a \times s) \bmod m$ ($a \neq 0$, $a \neq m$, m prime)

- For strings of characters of the form $s = s_0 s_1 \ldots s_{k-1}$ one can use:

$$h(s) = \left(\left(\sum_{i=0}^{k-1} B^i s_i \right) \bmod 2^w \right) \bmod m$$

e.g. B = 131 and w = word width (bits) of the computer (w = 32 or w = 64 is common).

Simple hash function

Choice of the hash function

- simple and quick computation
- even distribution of the data (example: compiler)

(Simple) division-residue method

$$h(k) = k \bmod m$$

How to choose m?

Examples:

a) m even $\rightarrow$ $h(k)$ even $\Leftrightarrow$ k even

Problematic if the last bit has a meaning (e.g. 0 = female, 1 = male)

b) $m = 2^p$ yields the p lowest dual digits of k

Rule: Choose m prime, and m is not a factor of any $r^i +/- j$,

where i and j are small, non-negative numbers and r is the radix of the representation.

Multiplicative method (1)

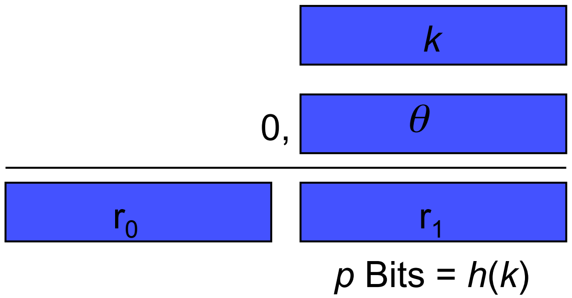
Choose constant $\theta, 0 < \theta < 1$

1. Compute $\quad k\theta \bmod 1 = k\theta - \lfloor k\theta \rfloor$

2. $h(k) = \lfloor m(k\theta \bmod 1) \rfloor$

Choice of *m* is uncritical, choose $m = 2^p$:

Computation of *h*(*k*) :



Multiplicative method (2)

Example:

$$\theta = \frac{\sqrt{5}-1}{2} \approx 0.6180339$$

$$k = 123456$$

$$m = 10000$$

$$\begin{aligned} h(k) &= \lfloor 10000(123456 * 0.61803\ldots \bmod 1) \rfloor \\ &= \lfloor 10000(76300{,}0041151\ldots \bmod 1) \rfloor \\ &= \lfloor 41.151\ldots \rfloor = 41 \end{aligned}$$

Of all numbers $0 \le \theta \le 1$, $\frac{\sqrt{5}-1}{2}$ leads to the most even distribution.

Universal hashing

Problem: if *h* is fixed → there are $S \subseteq U$ with many collisions

Idea of universal hashing:

Choose hash function *h* randomly

H finite set of hash functions

$$h \in H : U \to \{0,\ldots,m-1\}$$

Definition: *H* is universal, if for arbitrary $x,y \in U$:

$$\frac{\left|\{h \in H \mid h(x) = h(y)\}\right|}{|H|} \leq \frac{1}{m}$$

Hence: if x, $y \in U$, *H* universal, $h \in H$ picked randomly

$$\Pr_H(h(x) = h(y)) \leq \frac{1}{m}$$

Universal hashing

Definition:

$$\delta(x,y,h) = \begin{cases} 1, \text{ if } h(x) = h(y) \text{ and } x \neq y \\ 0, \text{ otherwise} \end{cases}$$

Extension to sets:

$$\delta(x,S,h) = \sum_{s \in S} \delta(x,s,h)$$

$$\delta(x,y,G) = \sum_{h \in G} \delta(x,y,h)$$

Corollary: *H* is universal, if for any $x,y \in U$

$$\delta(x,y,H) \leq \frac{|H|}{m}$$

A universal class of hash functions

Assumptions:

- $|U| = p$ (p prime) and $U = \{0, \ldots, p\text{-}1\}$
- Let $a \in \{1, \ldots, p\text{-}1\}$, $b \in \{0, \ldots, p\text{-}1\}$ and $h_{a,b} : U \rightarrow \{0,\ldots,m\text{-}1\}$ be defined as follows

$$h_{a,b} = ((ax+b) \bmod p) \bmod m$$

Then:

The set

$$H = \{h_{a,b} \mid 1 \leq a \leq p\text{-}1,\ 0 \leq b \leq p\text{-}1\}$$

is a universal class of hash functions.

Universal hashing - example

Hash table T of size 3, $|U| = 5$

Consider the 20 functions (set H):

x+0 2x+0 3x+0 4x+0

x+1 2x+1 3x+1 4x+1

x+2 2x+2 3x+2 4x+2

x+3 2x+3 3x+3 4x+3

x+4 2x+4 3x+4 4x+4

each (mod 5) (mod 3)

and the keys 1 und 4

We get:

(1*1+0) mod 5 mod 3 = 1 = (1*4+0) mod 5 mod 3

(1*1+4) mod 5 mod 3 = 0 = (1*4+4) mod 5 mod 3

(4*1+0) mod 5 mod 3 = 1 = (4*4+0) mod 5 mod 3

(4*1+4) mod 5 mod 3 = 0 = (4*4+4) mod 5 mod 3

Possible ways of treating collisions

Treatment of collisions:

- Collisions are treated differently in different methods.
- A data set with key s is called a colliding element if bucket $B_{h(s)}$ is already taken by another data set.
- What can we do with colliding elements?
  1. Chaining: Implement the buckets as linked lists. Colliding elements are stored in these lists.
  2. Open Addressing: Colliding elements are stored in other vacant buckets. During storage and lookup, these are found through so-called probing.