Universal hashing

Problem: if *h* is fixed → there are $S \subseteq U$ with many collisions

Idea of universal hashing:

Choose hash function *h* randomly

H finite set of hash functions

$$h \in H : U \rightarrow \{0,\ldots m-1\}$$

Definition: *H* is universal, if for arbitrary $x,y \in U$:

$$\frac{\{h \in H \mid h(x) = h(y)\}}{|H|} \leq \frac{1}{m}$$

Hence: if x, $y \in U$, H universal, $h \in H$ picked randomly

$$\Pr_H(h(x) = h(y)) \leq \frac{1}{m}$$

A universal class of hash functions

Assumptions:

- $|U| < p$ (p prime) and $U = \{0, \ldots, p\text{-}1\}$
- Let $a \in \{1, \ldots, p\text{-}1\}$, $b \in \{0, \ldots, p\text{-}1\}$ and $h_{a,b} : U \rightarrow \{0,\ldots,m\text{-}1\}$ be defined as follows

$$h_{a,b} = ((ax+b) \bmod p) \bmod m$$

Then:

The set

$$H = \{h_{a,b} \mid 1 \leq a \leq p\text{-}1,\ 0 \leq b \leq p\text{-}1\}$$

is a universal class of hash functions.

Universal hashing - example

Hash table *T* of size 3, |*U*| = 5

Consider the 20 functions (set *H*):

$x+0$	$2x+0$	$3x+0$	$4x+0$
$x+1$	$2x+1$	$3x+1$	$4x+1$
$x+2$	$2x+2$	$3x+2$	$4x+2$
$x+3$	$2x+3$	$3x+3$	$4x+3$
$x+4$	$2x+4$	$3x+4$	$4x+4$

each (mod 5) (mod 3) and the

key

s 1 und 4, let us consider the number of hash functions in H, such that h(1) = h(4).

a(1) +b	h'(1)=(a(1) +b) mod 5	a(4) +b	h'(4)=(a(4) +b) mod 5
1 2 3 4	1 2 3 4	4 8 12 16	4 3 2 1
2 3 4 5	2 3 4 0	5 9 13 17	0 4 3 2
3 4 5 6	3 4 0 1	6 10 14 18	1 0 4 3
4 5 6 7	4 0 1 2	7 11 15 19	2 1 0 4
5 6 7 8	0 1 2 3	8 12 16 20	3 2 1 0

Universal hashing - example

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x+1	2x+1	3x+1	4x+1
x+2	2x+2	3x+2	4x+2
x+3	2x+3	3x+3	4x+3
x+4	2x+4	3x+4	4x+4

each (mod 5) (mod 3) and the keys 1 und 4, let us consider the number of hash functions h in H, such that h(1) = h(4).

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2 3 4 5	2 3 4 0	5 9 13 17	0 4 3 2
3 4 5 6	3 4 0 1	6 10 14 18	1 0 4 3
4 5 6 7	4 0 1 2	7 11 15 19	2 1 0 4
5 6 7 8	0 1 2 3	8 12 16 20	3 2 1 0

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The set

$$H = \{h_{a,b} \mid 1 \leq a \leq p-1,\ 0 \leq b \leq p-1\}$$

is a universal class of hash functions.

$h_{a,b} = ((ax+b) \bmod p) \bmod m$

$H = \{h_{a,b} \mid 1 \leq a \leq p\text{-}1,\ 0 \leq b \leq p\text{-}1\}$ is a universal class of hash functions.

Proof

Consider two distinct keys x and y from $\{0,\ldots,p\text{-}1\}$, so that $x \neq y$. For a given hash function $h_{a,b}$, we let

$s = (ax + b) \bmod p$,

$t = (ay + b) \bmod p$.

Firstly, $s \neq t$ holds, since $s - t \equiv a(x - y) \pmod{p}$.

Possible ways of treating collisions

Treatment of collisions:

- Collisions are treated differently in different methods.
- A data set with key s is called a colliding element if bucket $B_{h(s)}$ is already taken by another data set.
- What can we do with colliding elements?
  1. Chaining: Implement the buckets as linked lists. Colliding elements are stored in these lists.
  2. Open Addressing: Colliding elements are stored in other vacant buckets. During storage and lookup, these are found through so-called probing.

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Theory I
Algorithm Design and Analysis

(6 Hashing: Chaining)

Prof. Th. Ottmann

Chaining (1)

- The hash table is an array (length *m*) of lists.
  Each bucket is realized by a list.

```
class hashTable {
    List[] ht;            // an array of lists
    hashTable (int m){              // Construktor
        ht = new List[m];
        for (int i = 0; i < m; i++)
            ht[i] = new List();   // Construct a list
    }
    ...
}
```

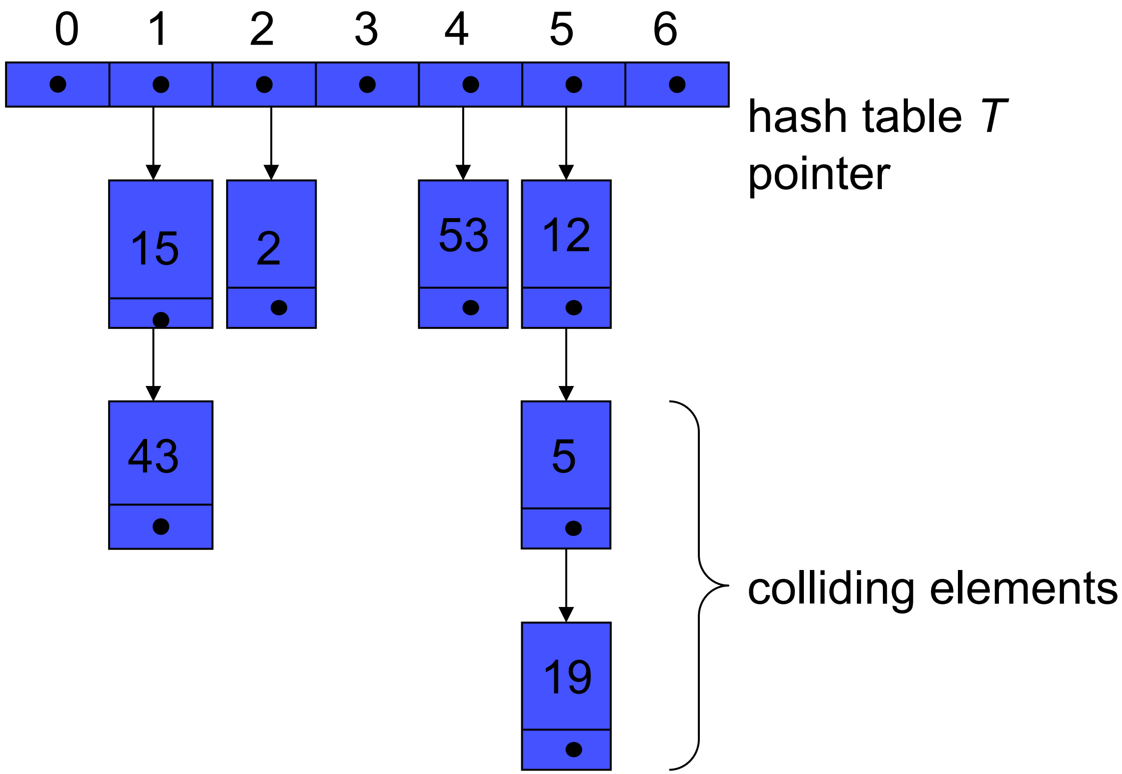
- Two different ways of using lists:
  1. Direct chaining:
  Hash table only contains list headers; the data sets are stored in the lists.

- 2. Separate chaining:
  Hash table contains at most one data set in each bucket as well as a list header.
  Colliding elements are stored in the list.

Hashing by chaining

Keys are stored in overflow lists

$h(k) = k \bmod 7$



This type of chaining is also known as direct chaining.

Chaining

Lookup key *k*

- Compute $h(k)$ and overflow list $T[h(k)]$
- Look for *k* in the overflow list

Insert a key *k*

- Lookup *k* (fails)
- Insert *k* in the overflow list

Delete a key *k*

- Lookup *k* (successfully)
- Remove *k* from the overflow list

→ only list operations

Analysis of direct chaining

Uniform hashing assumption:

- All hash addresses are chosen with the same probability, i.e.:

  $Pr(h(k_i) = j) = 1/m$

- independent from operation to operation

Average chain length for n entries:

$$n/m = \alpha$$

Definition:

C'_n = Expected number of entries inspected during a failed search

C_n = Expected number of entries inspected during a successful search

Analysis:

$$C'_n = \alpha$$

$$C_n \approx 1 + \frac{\alpha}{2}$$

Chaining

Advantages:

+ C_n and C'_n are small
+ $\alpha > 1$ possible
+ real distances
+ suitable for secondary memory

Efficiency of lookup

α	C_n (successful)	C'_n (fail)
0.50	1.250	0.50
0.90	1.450	0.90
0.95	1.457	0.95
1.00	1.500	1.00
2.00	2.000	2.00
3.00	2.500	3.00

Disadvantages:

- Additional space for pointers
- Colliding elements are outside the hash table

Summary

Analysis of hashing with chaining:

- worst case:
  $h(s)$ always yields the same value, all data sets are in a list.
  Behavior as in linear lists.

- average case:
  – Successful lookup & delete: complexity (in inspections) ≈ 1 + 0.5 × load factor
  – Failed lookup & insert: complexity ≈ load factor

  This holds for direct chaining, with separate chaining the complexity is a bit higher.

- best case:
  lookup is an immediate success: complexity $\in O(1)$.